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ON THE RELATIVE EQUILIBRIA OF A SATELLITE-GYROSTAT, THEIR BRANCHINGS AND STABILITY*

V.N. RUBANOVSKII

The set of relative equilibria of a satellite-gyrostat in a Newtonian gravitational field is studied. The simple geometrical form of this set is described. The branching and stability of the equilibria of a symmetric gyrostat are considered. The results are represented by bifurcation diagrams, on which the degree of stability of the equilibria is distributed in accordance with a law whereby the stability changes at a fixed value of the gyrostatic moment.

1. In some problems of gyrostat dynamics in a Newtonian gravitational field /1-6/, the determination of the positions of relative equilibrium of the gyrostat amounts to finding the stationary values of the function

$$W = \frac{1}{2}\sum_{j=1}^{3}(3hA_j\gamma_j^2 - A_j\beta_j^2 - 2k_j\beta_j)$$

under the conditions

$$\pi_\gamma = \gamma_1^2+\gamma_2^2+\gamma_3^2-1=0,\ \pi_\beta = \beta_1^2+\beta_2^2+\beta_3^2-1=0 \tag{1.1}$$
$$\pi_{\gamma\beta} = \gamma_1\beta_1+\gamma_2\beta_2+\gamma_3\beta_3 - 0$$

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Here, $A_1 \leqslant A_2 \leqslant A_3$ are the principal central moments of inertia of the gyrostat, k_j are the projections onto the principal axes x_j of the central ellipsoid of inertia of the vector $\mathbf{k}$, which is proportional to the vector of the gyrostatic moment $\mathbf{g}$.

In particular, for problems of the relative equilibrium of a satellite-gyrostat in a Kepler circular orbit /1-4/ $h = 1, \mathbf{k} = \mathbf{g}\omega^{-1}$, ω is the orbital angular velocity, and γ_j and β_j are the projections onto the x_j axes of the unit vectors along the radius vector and the binormal of the orbit.

The equations of relative equilibrium can be written as /4-6/

$$\begin{aligned} &\partial W_*/\partial\gamma_1 = 3h\,[(A_1-\sigma)\,\gamma_1 + \lambda\beta_1] = 0 \quad (1\ 2\ 3) \\ &\partial W_*/\partial\beta_1 = 3\lambda h\gamma_1 + (\nu - A_1)\,\beta_1 - k_1 = 0 \quad (1\ 2\ 3) \\ &2W_* = 2W + 6\lambda h\pi_{\gamma\beta} + \nu\pi_\beta - 3\sigma h\pi_\gamma \end{aligned} \tag{1.2}$$

where λ, σ and ν are the undetermined Lagrange multipliers.

We fix λ, σ, ν, and solve Eqs.(1.2) for γ_j, β_j:

$$\begin{aligned} &\gamma_1 = \lambda k_1\Phi_1^{-1}, \quad \beta_1 = (\sigma - A_1)\,k_1\Phi_1^{-1} \quad (1\ 2\ 3) \\ &\Phi_1 = 3h\lambda^2 + (\sigma - A_1)(\nu - A_1) \quad (1\ 2\ 3) \end{aligned} \tag{1.3}$$

Substituting the values (1.3) into (1.1), we obtain a system of three linear equations for k_1^2, k_2^2, k_3^2, from which, under the condition

$$\lambda A \neq 0, \quad A = (A_2 - A_3)(A_3 - A_1)(A_1 - A_2)$$

we obtain

$$k_1^2 = \frac{(A_3 - A_2)\,L_1\Phi_1^2}{\lambda^2 A}, \quad L_1 = \lambda^2 + (\sigma - A_2)(\sigma - A_3) \quad (1\ 2\ 3) \tag{1.4}$$

Using (1.4), we can write (1.3) as /6/

$$\gamma_1^2 = \frac{(A_3 - A_2)\,L_1}{A}, \quad \beta_1^2 = \frac{(A_3 - A_2)(\sigma - A_1)^2 L_1}{\lambda^2 A} \quad (1\ 2\ 3) \tag{1.5}$$

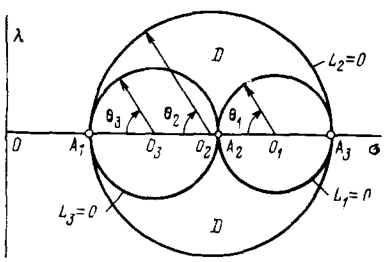

Fig.1

For the geometrical representation of the set of all relative equilibria (1.3) and (1.4), we consider in the space of parameters λ, σ, ν the domain /6/ D, defined by the inequalities $L_1 > 0, L_2 < 0, L_3 > 0$. To points of D there corresponds real values of γ_j, β_j, k_j. The domain D is a cylindrical solid, whose profile (Fig.1) is formed by the three circles $L_j = 0$ $(j = 1, 2, 3)$, which are similar to the Moire circles familiar in the theory of elasticity. It follows from (1.5) that the orientation of the gyrostat body in relative equilibrium is independent of the parameter ν. To each point of the profile of D there correspond eight positions of equilibrium, for which γ_j, β_j $(j = 1, 2, 3)$ correspond to eight different combinations of signs of the k_j. To points which are symmetrical about the plane $\lambda = 0$ there correspond dynamically equivalent equilibrium positions which differ by a 180° rotation about the vector $\boldsymbol{\beta}$. To the generators $\lambda = 0, \sigma = A_j$ $(j = 1, 2, 3)$ of the boundary ∂D of domain D, along which we have mutual contact of each two of the three cylinders $L_j = 0$ $(j = 1, 2, 3)$, there correspond the families of equilibria /3, 4/

$$\begin{aligned} &\gamma_1 = 1, \ \gamma_2 = \gamma_3 = \beta_1 = 0, \ \beta_2 = \sin\theta, \ \beta_3 = \cos\theta \quad (1\ 2\ 3) \\ &k_1 = 0, \ k_2 = (\nu - A_2)\sin\theta, \ k_3 = (\nu - A_3)\cos\theta \quad (0 \leqslant \theta < 2\pi) \end{aligned} \tag{1.6}$$

In the equilibrium positions (1.6), the x_1 axis is parallel to the vector $\boldsymbol{\gamma}$, and x_2 and x_3 axes are perpendicular to the vector $\boldsymbol{\gamma}$, while the x_3 axis is at an angle θ to the vector $\boldsymbol{\beta}$, and the rotor axis is orthogonal to the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$.

To points of the cylinders $L_j = 0$ $(j = 1, 2, 3)$ there correspond the families of equilibria /3, 4/

$$\begin{aligned} &\gamma_1 = \beta_1 = 0, \ \gamma_2 = \beta_3 = \cos{}^1/_2\theta_1, \ \gamma_3 = -\beta_2 = -\sin{}^1/_2\theta_1 \quad (1\ 2\ 3) \\ &k_2 = [\nu - A_2 - 3h\,(A_3 - A_2)\cos^2{}^1/_2\theta_1]\sin{}^1/_2\theta_1 \quad (0 \leqslant \theta_1 < 2\pi) \\ &k_3 = [\nu - A_3 + 3h\,(A_3 - A_2)\sin^2{}^1/_2\theta_1]\cos{}^1/_2\theta_1, \ k_1 = 0 \end{aligned} \tag{1.7}$$

For the solution (1.7), the x_1 axis is collinear with the vector $\boldsymbol{\alpha} = \boldsymbol{\beta} \times \boldsymbol{\gamma}$, directed along the tangent to the orbit towards the motion of the gyrostat centre of mass, the x_2 and x_3 axes are perpendicular to the vector $\boldsymbol{\alpha}$, while the x_3 axis makes an angle ${}^1/_2\theta_1$ to the vector $\boldsymbol{\beta}$, and the rotor axis is orthogonal to the vector $\boldsymbol{\alpha}$. As $A_2 \to A_1$ $(A_2 \to A_3)$, the cylinder $L_3 = 0$ $(L_1 = 0)$ stretches out continuously into the straight line $\lambda = 0, \sigma = A_1$ $(\lambda = 0, \sigma = A_3)$, and the domain D degenerates into the surface of the cylinder $L_2 = 0$, while the families (1.6), (1.7) exhaust all the relative equilibria of the gyrostat.

2. Sufficient conditions for stability of the relative equilibria (1.3), (1.4) are obtained in /4/. These conditions can be written in terms of the parameters λ, σ, ν, as

$$\begin{aligned} &a > 0,\ \Delta' = 2a\nu + b > 0,\ \Delta = a\nu^2 + b\nu + c > 0 \\ &a = \lambda^{-2}H,\ b = 3hH' - 2\sigma\lambda^{-2}H - \lambda^{-4}H^2 \\ &H = (\sigma - A_1)(\sigma - A_2)(\sigma - A_3),\ H' = dH/d\sigma \\ &c = {}^9/_2 h^2\lambda^2 H'' + 3h[(3h-1)H - \sigma H'] + (\sigma^2 - 3hH')\lambda^{-2}H + \\ &\quad \sigma\lambda^{-4}H^2 \\ &\Delta = \lambda^{-2}H(\nu - \sigma)^2 + (3hH' - \lambda^{-4}H^2)(\nu - \sigma) + 3h[{}^3/_2 h\lambda^2 H'' + \\ &\quad (3h-1)H - \lambda^{-2}H'H] \end{aligned} \tag{2.1}$$

We take the two surfaces in the space of parameters λ, σ, ν,

$$\nu = \nu^{\pm}(\lambda, \sigma),\ \nu^{\pm} = (b \pm \sqrt{b^2 - 4ac})/(2a)$$

defined by the equation $\Delta = 0$. The functions $\nu = \nu^{\pm}$ take real values for all admissible values $\lambda \neq 0, \sigma$. The surface $\nu = \nu^+$ cuts the cylinders $L_j = 0$ along the curves G_j, whose projections onto the plane $\lambda = 0$ are the hyperbolas

$$\nu = A_1 + 3h(\sigma - A_2)(\sigma - A_3)(\sigma - A_1)^{-1} \quad (1\ 2\ 3)$$

At the same time, the curve G_j is the line of intersection of the cylinder $L_j = 0$ and the cone $\Phi_j = 0$. The surface $\nu = \nu^-$ cuts the cylinders $L_j = 0$ along the ellipses E_j, which are located in parallel planes and project onto the plane $\lambda = 0$ as the segments of parallel straight lines

$$\nu = (1 + 6h)\sigma - 3h(A_2 + A_3) \qquad (1\ 2\ 3)$$

The surface $\nu = \nu^+$ has a discontinuity at $\sigma = A_2$. As $\sigma \to A_2$, it tends asymptotically to the plane $\sigma = A_2$. With $\sigma < A_2$, the surfaces $\nu = \nu^{\pm}$ intersect only under the condition $3h > (A_3 - A_2)(A_2 - A_1)^{-1}$, the their lines of intersection are then located between the cylinders $L_2 = 0, L_3 = 0$, while their ends are on the cylinder $L_3 = 0$. With $\sigma > A_2$, the surface $\nu = \nu^{\pm}$ always intersect; their line of intersection is then between the cylinders $L_1 = 0, L_2 = 0$, while its ends lie on the cylinder $L_1 = 0$.

Conditions (2.1) are equivalent to

$$a > 0,\ \nu > \nu_2,\ \nu_1 = \min(\nu^+, \nu^-),\ \nu_2 = \max(\nu^+, \nu^-) \tag{2.2}$$

We see from (2.2) that the equilibria for which $\nu > \nu_2, \nu_1 < \nu < \nu_2, \nu < \nu_1$, have a degree of instability χ, which is respectively equal to $0, 1, 2$, if $a > 0$, and to $1, 2, 3$, if $a < 0$.

3. Let us study the relative equilibria of the symmetric gyrostat under the conditions

$$A_1 = A_2 < A_3,\ (e_1^2 + e_2^2)e_3 \neq 0 \tag{3.1}$$

where e_j are the projections onto the x_j axes of the unit vector in the direction of the vector $\mathbf{k}$, $k_j = ke_j$ $(j = 1, 2, 3)$, where k is a variable parameter.

Under conditions (3.1), Eqs. (1.2) and (1.1) have two one-parameter families of solutions:

$$\begin{aligned} &\gamma_1 = \lambda k e_1 \Phi_1^{-1},\ \beta_1 = ke_1(\sigma - A_1)\Phi_1^{-1}\ (1\ 2\ 3) \\ &\Phi_1 = \Phi_2 = (\sigma - A_1)[\nu - A_1 - 3h(\sigma - A_3)],\ \lambda^2 = (\sigma - \\ &\quad A_1)(A_3 - \sigma) \\ &\Phi_3 = (\sigma - A_3)[\nu - A_3 - 3h(\sigma - A_1)],\ e_1^2 + e_2^2 = e_3^2\varkappa^3 \\ &e_3^2(A_3 - A_1)(\sigma - A_1)\varkappa^3 k^2 = \Phi_1^2,\ e_3^2(A_3 - A_1)(A_3 - \\ &\quad \sigma)k^2 = \Phi_3^2 \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\alpha_1 = \frac{e_2 k}{\varkappa^{3/2}(\nu - A_3)},\quad \alpha_2 = \frac{e_1 k}{\varkappa^{3/2}(\nu - A_3)},\quad \alpha_3 = \frac{e_3\varkappa^{3/2}k}{A_1 - \nu} \\ &\beta_1 = \frac{e_1 k}{\nu - A_1},\quad \beta_2 = \frac{e_2 k}{\nu - A_1},\quad \beta_3 = \frac{e_3 k}{\nu - A_3} \\ &\lambda = 0,\quad \sigma = A_1,\quad k^2 = \frac{(\nu - A_1)^2(\nu - A_3)^2}{[(\nu - A_1)^2 + \varkappa^3(\nu - A_3)^2]e_3^2} \end{aligned} \tag{3.3}$$

The equilibrium positions (3.2) and (3.3) can be represented geometrically in the space of parameters k, σ, ν, by points of the curve Γ, whose branches Γ_1 and Γ_2 are given by the last two equations in (3.2) and (3.3).

Let us study the branches Γ_1 and Γ_2, corresponding to the equilibria (3.2) and (3.3). We start with the curve Γ_1.

Dividing the last two equations in (3.2) term by term, and introducing the auxiliary parameter μ, we obtain

$$\frac{\nu - A_1 - 3h(\sigma - A_3)}{\nu - A_3 - 3h(\sigma - A_1)} = \mu, \quad \mu = \varkappa\sqrt{\frac{(A_3-\sigma)\varkappa}{\sigma - A_1}} \quad (-\infty<\mu<\infty) \tag{3.4}$$

From (3.4) and (3.2), we have

$$\nu = \frac{(A_1 - A_3\mu)(\mu^2+\varkappa^3) + 3h(A_1 - A_3)(\mu + \varkappa^3)\mu}{(1-\mu)(\mu^2+\varkappa^3)}, \quad \sigma = \frac{A_1\mu^2 + A_3\varkappa^3}{\mu^2+\varkappa^3} \tag{3.5}$$

We now add the last two equations in (3.2) term by term, and using (3.4), we obtain

$$(A_3 - A_1)k^2 = [(\sigma - A_1)\mu^2 + A_3 - \sigma][\nu - A_3 - 3h(\sigma - A_1)]^2$$

On substituting for ν and σ from (3.5), we now finally obtain

$$k^2 = \frac{(1+3h)^2(A_3-A_1)^2\mu^2}{e_3^2(1-\mu)^2(\mu^2+\varkappa^3)} \tag{3.6}$$

In short, in the space of parameters k, σ, ν, the branch Γ_1 of curve Γ is a spatial curve whose parametric form is given by (3.5) and (3.6).

In Fig.2 we plot the function given by Eq.(3.6), and in Fig.3 (the continuous curve) the projection of the curve Γ_1 onto the $\sigma = 0$ plane. Here,

$$k^{*2} = \frac{(1+3h)^2(A_3-A_1)^2}{(1+\varkappa)^3 e_3^2}, \quad \nu^* = \frac{A_1 + A_3\varkappa + 3h(A_1-A_3)(1-\varkappa)}{1+\varkappa},$$
$$\mu^* = -\varkappa$$

Hence we conclude that, for the values $0 < k^2 < k^{*2}$ and $k^2 > k^{*2}$, there are respectively four and two positions of relative equilibrium of the gyrostat, which are given by (3.2).

For equilibria (3.2), the stability conditions (2.1) take the form

$$a = \frac{(A_1 - A_3)\varkappa^3}{\mu^2+\varkappa^3} > 0, \quad \Delta' = \frac{a(1+3h)(A_1-A_3)[(1+\mu)\varkappa^3 + 2\mu^3]}{(1-\mu)(\mu^2+\varkappa^3)} > 0 \tag{3.7}$$
$$\Delta = {}^1/_2\varkappa^3 e_3^2(A_1 - A_3)(1-\mu)\,dk^2/d\mu > 0$$

Now take the branch Γ_2, corresponding to equilibria (3.3). In the space of parameters k, σ, ν, it is a plane curve which lies in the plane $\sigma = A_1$ and is defined by the last of Eqs. (3.3).

In Fig.3 (the broken curve) we show the projection of the curve Γ_2 onto the plane $\sigma = 0$. Here,

$$k_*^2 = \frac{(A_3-A_1)^2}{(1+\varkappa)^3 e_3^2}, \quad \nu_* = \frac{A_1 + A_3\varkappa}{1+\varkappa}, \quad \frac{k^{*2}}{k_*^2} = (1+3h)^2 > 1$$
$$\nu_* - \nu^* = 3h(A_3 - A_1)(1-\varkappa)^{-1}(1+\varkappa)^{-1}$$

Hence we see that, for the values $0 < k^2 < k_*^2$ and $k^2 > k_*^2$, there are respectively four and two positions of relative equilibrium, given by (3.3).

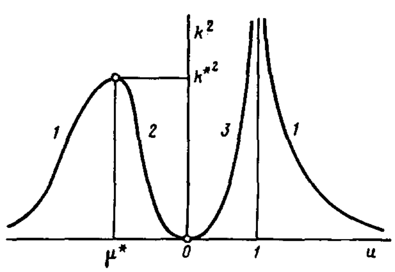

Fig.2

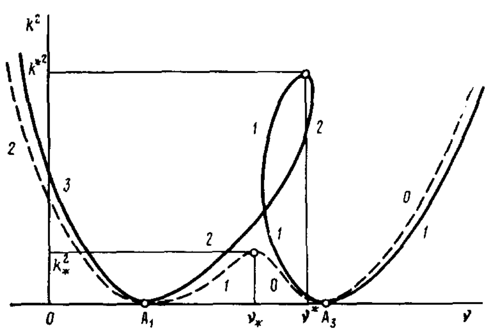

Fig.3

The sufficient conditions for stability of the relative equilibria (3.3) are given by (2.1), where we now have /4/

$$a = (A_3 - A_1)\alpha_3^2, \; b = (A_1 - A_3)\alpha_3^2[(A_1 + A_3)(1 - \beta_3^2) + 2A_1\beta_3^2]$$
$$c = (A_3 - A_1)A_1\alpha_3^2[A_1\beta_3^2 + A_3(1-\beta_3^2)]$$

Using these realtions and (3.3), we can write conditions (2.1) as

$$a = \frac{(A_3 - A_1)(\nu - A_3)^2 \varkappa^3}{(\nu - A_1)^2 + \varkappa^3 (\nu - A_3)^2} > 0, \tag{3.8}$$
$$\Delta' = \frac{a(2\nu - A_3 - A_1)(\nu - A_3)^2 \varkappa^3 + 2(\nu - A_1)^2}{(\nu - A_1)^2 + \varkappa^3 (\nu - A_3)^2} > 0$$
$$\Delta = {}^1/_2 \varkappa^3 e_3^2 (A_3 - A_1)(\nu - A_3)\, dk^2/d\nu > 0$$

4. Let us study the stability of equilibria (3.2) and (3.3). Let χ be the degree of instability (DI). The equilibria for which $\chi = 0$ are stable, while when $\chi = 1$ or $\chi = 3$, they are unstable. No answer is given by Routh's theorem when $\chi = 2$. If this equilibrium is stable, its stability is of a gyroscopic type, and by the Kelvin-Chetayev theorem /7/, it is temporary and is destroyed under the action of a system of dissipative forces with total dissipation.

From (3.7), (3.6) and Fig.2, we see that, for the equilibria (3.2), $a < 0$ and

$$\Delta' < 0,\ \Delta < 0,\ \chi = 1,\ \text{if}\ \mu < \mu^*\ \text{or}\ \mu > 1$$
$$\Delta' > 0,\ \Delta > 0,\ \chi = 2,\ \text{if}\ \mu^* < \mu < 0$$
$$\Delta' > 0,\ \Delta < 0,\ \chi = 3,\ \text{if}\ 0 < \mu < 1$$

For the equilibria (3.3), we see from (3.8), the last relation in (3.3), and Fig.3 (broken curve), that $a > 0$ and

$$\Delta' < 0,\ \Delta > 0,\ \chi = 2,\ \text{if}\ \nu < A_1$$
$$\Delta' < 0,\ \Delta < 0,\ \chi = 1,\ \text{if}\ A_1 < \nu < \nu_*$$
$$\Delta' > 0,\ \Delta > 0,\ \chi = 0,\ \text{if}\ \nu_* < \nu < A_3\ \text{or}\ \nu > A_3$$

The results of analysing the stability conditions for equilibria (3.2) and (3.3) are shown in Figs.2 and 3, where the numbers 0, 1, 2, 3 on the branches of the curve Γ indicate the DI of the respective equilibria. Notice that, in Fig.3, the DI distribution on the branches of the curve is subject to the law whereby the stability changes /7/ at a fixed value of the parameter k; in particular, the DI only changes at points of bifurcation.

Similar bifurcation diagrams can be plotted for equilibria (3.2) and (3.3) in the case when $A_1 = A_2 > A_3$.

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